

## Note

# Energy Conserving Galerkin Finite Element Schemes for the Primitive Equations of Numerical Weather Prediction

### 1. INTRODUCTION

For extended time integrations of the equations of atmospheric motion the importance of observing energy and other conservation properties of the finite difference equations has been recognised. The schemes in use are based on ideas developed in [1]. The use of energy conserving schemes generally improves the nonlinear stability of the schemes and therefore reduces the need for intentional numerical diffusion. For climate models, conservation properties of numerical approximations may be a desirable property in itself.

For the purposes of medium range weather prediction, it may be desirable to refine the grid in certain areas of particular interest for the forecast in the target area. Such models with grid refinement may suffer from nonlinear instability. Consequently short range forecast models of this type are usually implemented with relaxation procedures, as proposed in [10]. Since conservation properties of schemes are known to enhance the nonlinear stability it is interesting to formulate conserving schemes for irregular grids.

Finite element schemes can be supposed to have a good nonlinear stability because of the reduced aliasing error of the Galerkin method. Furthermore, they automatically transfer their conservation properties to the case of irregular resolution. Standard finite element schemes can be expected to conserve quadratic moments, from which energy conservation for the nondivergent flow follows (see [2, 3]). Also, the semidiscretized equations, obtained by approximating only the vertical coordinate in  $\sigma$ -coordinate equations and considering the horizontal coordinates as continuous, are normally energy conserving (see [7]). Again in this case the energy is a second order moment.

Conserving schemes, including those with energy conservation were formulated in [4] for the Boussinesq equations. A relation between the approximation spaces for pressure and temperature is necessary to obtain energy conservation. In particular, linear elements for the velocities and pressure are not admitted in [4] for an energy conserving scheme. In [11], similar schemes were investigated, which allow, however, a less restricted choice of basis functions.

This paper presents some energy conserving finite element schemes for the primitive meteorological equations. Schemes are formulated for the two-dimensional shallow water equations. All schemes presented can easily be generalized to

energy conserving schemes of the  $\sigma$ -system equations. This may be achieved by starting from the semidiscretized equations given in [7]. One of the schemes requires, as in [4], a relation between the approximation spaces for pressure and temperature. Another scheme uses only linear elements for all fields, and therefore may be a good candidate for implementation in large models.

Since the schemes use only general relations between the basis function spaces, the developments are valid for other Galerkin schemes, such as the spectral method or the parameter fluid models proposed in [5, 6]. Questions of practical implementation are, however, discussed only for the use of linear finite elements for the velocity field.

## 2. GALERKIN PROJECTION

For fields  $\phi$ , we assume the following representation:

$$\tilde{\phi}(\mathbf{r}) = \sum_{v=1}^N \phi_v e_v(\mathbf{r}). \tag{1}$$

In Eq. (1),  $\mathbf{r}$  is the vector  $x, y$ , with  $x, y$  being horizontal coordinates. Depending on the choice of the basis functions  $e_v$ , different approximating spaces  $\Omega_1, \Omega_2, \dots$ , are obtained. We consider here only linear and quadratic finite element (FE) spaces, though this specialisation is used only to discuss questions of practical implementation.

To formulate FE schemes for prognostic equations, it is necessary to approximate a rather general field  $\phi(\mathbf{r})$  by a function  $\tilde{\phi}(\mathbf{r})$  of the space given by Eq. (1). This is achieved by the Galerkin projection, where the coefficients  $\phi_v$  in Eq. (1) are defined by

$$(e_\xi, \tilde{\phi}) = (e_\xi, \phi), \quad \xi \in \{1, \dots, N\}. \tag{2}$$

The scalar product  $(a, b)$  is defined as

$$(a, b) = \int a(\mathbf{r}) b(\mathbf{r}) d\mu. \tag{3}$$

Here, it is sufficient to assume the measure  $d\mu$  to be of the form

$$d\mu = w(\mathbf{r}) dx dy, \tag{4}$$

with  $w$  being a positive continuous function. The relation between the fields  $\phi$  and  $\tilde{\phi}$ , given by Eq.(2), can be written as

$$\tilde{\phi} = G\phi \tag{5}$$

with  $G$  being the Galerkin projection operator.

The following properties of  $G$  can easily be verified:

$$\begin{aligned} G^2 &= G \cdot G = G, \\ (Ga, b) &= (a, Gb), \\ G(a\phi_2 + b\phi_2) &= aG\phi_1 + bG\phi_2. \end{aligned} \tag{6}$$

By choosing different spaces of basis functions, and different  $w$  in Eq. (4), different Galerkin projections  $G_1, G_2, \dots$ , belonging to scalar products  $(\cdot, \cdot)_1$  and  $(\cdot, \cdot)_2, \dots$  etc., can be defined.

For prognostic equations of the form

$$\dot{\phi} = f(\phi) \tag{7}$$

the approximation

$$\dot{\tilde{\phi}} = Gf(\tilde{\phi}) \tag{8}$$

is referred to as the standard FE-scheme. Of course, in Eq. (8), a different  $G$  can be used for different components of  $\phi$ .

### 3. ENERGY CONSERVING SCHEMES FOR THE SHALLOW WATER EQUATIONS

The schemes are presented here for the  $x$ - $y$  plane. The generalization to the rotating sphere is obvious. An energy conserving scheme for the shallow water equations is

$$\begin{aligned} \dot{u} &= -G_1(uu_x + vu_y + H_x), \\ \dot{v} &= -G_1(uv_x + vv_y + H_y), \\ \dot{H} &= -G_2((uH)_x + (vH)_y). \end{aligned} \tag{9}$$

The above scheme will be referred to as scheme A. Let  $\Omega_1$  and  $\Omega_2$  be the spaces belonging to the projections  $G_1, G_2$  and  $(\cdot, \cdot)_1$  and  $(\cdot, \cdot)_2$  be the corresponding scalar products. The conditions of energy conservation require a relation between the spaces  $\Omega_1$  and  $\Omega_2$  as well as special definitions for the scalar products.

$$\begin{aligned} u, v &\in \Omega_2, \\ H &\in \Omega_2, \\ \Omega_1^2 &\subset \Omega_2, \end{aligned} \tag{10}$$

$$(a, b)_1 = \int H(r) a(\mathbf{r}) b(\mathbf{r}) dx dy, \tag{11}$$

$$(a, b)_2 = \int a(r) b(r) dx dy. \tag{12}$$

The quadratic space  $\Omega_1^2$  of  $\Omega_1$  is defined as the space generated according to Eq. (1) by all products  $e_v(\mathbf{r}) e_\mu(\mathbf{r})$ , where the  $e_v$  form a basis of  $\Omega_1$ .

In one space dimension, the quadratic space of the space of linear splines is the space of quadratic splines with the same node points. In two space dimensions, not every quadratic space can be used. For example, the quadratic element space defined in [8] is not the quadratic space of the corresponding linear element scheme. For a regular rectangular mesh the grid for the definition of the quadratic space of the linear splines is given in Fig. 1. Let  $e_v(x)$  be the basis functions of linear splines in one space dimension, and let the basis functions of linear splines in two space dimensions be given as

$$b_{v,\mu}^1(x, y) = e_v(x) e_\mu(y). \tag{13}$$

Then the following basis functions are associated with the node points in Fig. 1 for its quadratic space:

$$b_{v,\mu}^2 = b_{v,\mu}^1, \tag{14}$$

$$b_{v+(1/2),\mu}^2 = e_v(x) e_{v+1}(x) e_\mu(y), \tag{15}$$

$$b_{v,\mu+(1/2)}^2 = e_v(x) e_\mu(y) e_{\mu+1}(y), \tag{16}$$

$$b_{v+(1/2),\mu+(1/2)}^2 = e_v(x) e_{v+1}(x) e_\mu(y) e_{\mu+1}(y). \tag{17}$$

The basis function  $b_{v+(1/2),\mu+(1/2)}^2$  was missed out in the quadratic spline space defined in [8]. The condition given by Eq. (10) is rather similar to a condition used in [4] to obtain energy conservation. In [4] the observation has already been made that not every space of quadratic splines for the pressure field leads to energy conservation.

To prove the energy conservation of Eqs. (9), form the time derivative of the total energy and denote the vector  $u, v$  by  $\mathbf{u}$ . We assume that the FE discretization of  $\mathbf{u}$

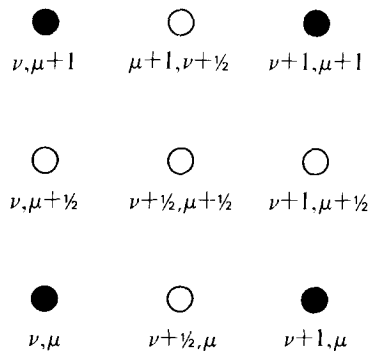


FIG. 1. The grid for the definition of the quadratic space of linear splines for regular rectangular resolution.

implies  $\mathbf{u} \cdot \mathbf{n} = 0$  at the boundaries of the region, with  $n$  being the normal vector of the boundary.

$$\begin{aligned} \dot{E} &= \frac{d}{dt} \int \frac{1}{2} H(\mathbf{u}^2 + H) dx dy \\ &= \frac{1}{2} (\dot{H}, \mathbf{u}^2)_2 + (\dot{H}, H)_2 + (\mathbf{u}, \dot{\mathbf{u}})_1. \end{aligned} \quad (18)$$

The definition equations (11) and (12) have been used in Eq. (18). According to Eqs. (9) we obtain

$$\begin{aligned} \dot{E} &= -(G_2 \operatorname{div}(H\mathbf{u}), \frac{1}{2}\mathbf{u}^2 + H)_2 \\ &\quad - (\mathbf{u}, G_1(\mathbf{u} \cdot \operatorname{grad} \mathbf{u} + \operatorname{grad} H))_1, \end{aligned} \quad (19)$$

Using Eq. (6) for the first term of Eq. (19), and transforming  $\operatorname{grad} \mathbf{u}$  term in the second, we obtain

$$\begin{aligned} \dot{E} &= -(\operatorname{div}(H\mathbf{u}), G_2(\frac{1}{2}\mathbf{u}^2 + H)) \\ &\quad - (\mathbf{u}, G_1 \operatorname{grad}(\frac{1}{2}\mathbf{u}^2 + H))_1 \\ &\quad - (\mathbf{u}, G_1 \mathbf{u} \times \operatorname{curl} \mathbf{u})_1. \end{aligned}$$

The last term of this equation is 0 by Eq. (6).

Using Eqs. (6) and (10), one can eliminate the Galerkin projections in Eq. (19),

$$\begin{aligned} \dot{E} &= -(\operatorname{div} H\mathbf{u}, \frac{1}{2}\mathbf{u}^2 + H)_2 \\ &\quad - (uH, (\frac{1}{2}\mathbf{u}^2 + H)_x)_2 - (vH, (\frac{1}{2}\mathbf{u}^2 + H)_y)_2. \end{aligned} \quad (20)$$

Performing a partial integration in Eq. (20), using the condition that  $\mathbf{u} \cdot \mathbf{n}$  is 0 at the boundaries, one can see that the energy is conserved.

For easy practical implementation it may be convenient to have a scheme which allows linear finite element spaces for the approximation of all fields. Such schemes can be obtained using a form of the shallow water equations used in [9] to obtain difference schemes with conservation properties. It will be referred to as scheme B;

$$\begin{aligned} \dot{u} &= G_1(\tilde{\eta}v - (G_2(\frac{1}{2}\mathbf{u}^2 + H))_x), \\ \dot{v} &= G_1(-\tilde{\eta}u - (G_2(\frac{1}{2}\mathbf{u}^2 + H))_y), \\ \dot{H} &= -G_2 \operatorname{div}(H\mathbf{u}), \\ \eta &= v_x - u_y. \end{aligned} \quad (21)$$

Again we require that  $G_1$  corresponds to the scalar product given in Eq. (11) and  $G_2$  corresponds to that of Eq. (12).  $\tilde{\eta}$  can be interpolated arbitrarily from  $\eta$ . No condition on the approximation spaces belonging to  $G_1$  and  $G_2$  is imposed.

To prove energy conservation for Eqs. (21), we consider the energy equation

$$\begin{aligned} \dot{E} &= \frac{d}{dt} \int H \cdot \frac{1}{2} (\mathbf{u}^2 + H) \, dx \, dy \\ &= - \left( \mathbf{u}, G_1 \left( \text{grad } G_2 \left( \frac{1}{2} \mathbf{u}^2 + H \right) \right) \right)_1 - \left( G_2 \text{div } \mathbf{u} H, \frac{1}{2} \mathbf{u}^2 + H \right)_2. \end{aligned}$$

Using Eqs. (6) and performing a partial integration, observing the boundary conditions for  $\mathbf{u}$ , we obtain:

$$\dot{E} = -(\text{div } H\mathbf{u}, G_2(\frac{1}{2}\mathbf{u}^2 + H))_2 + (G_2 \text{div } \mathbf{u}H, \frac{1}{2}\mathbf{u}^2 + H)_2. \quad (22)$$

Using again Eq. (6), we see that the right-hand side of Eq. (22) is zero.

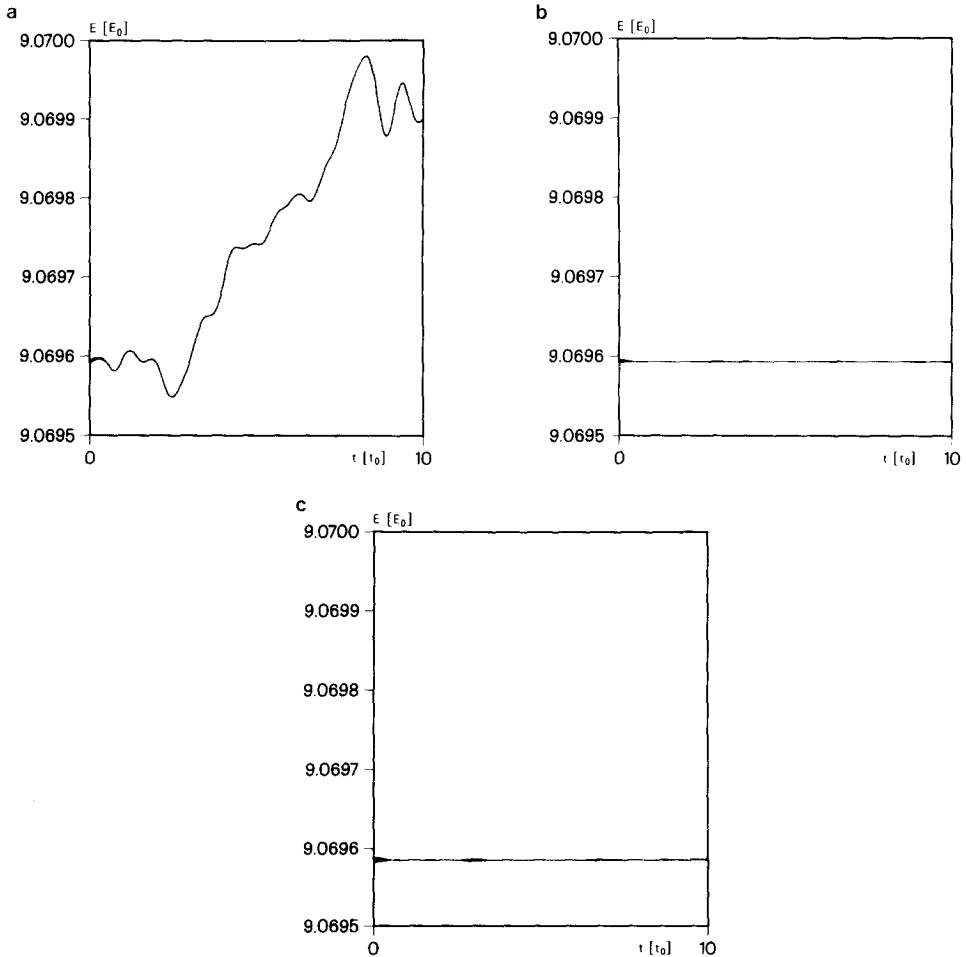


FIG. 2. Energy diagram with different schemes for a one dimensional gravity wave: (a) control, (b) scheme A, (c) scheme B.

## 4. A NUMERICAL CALCULATION

The possible impact of energy conservation on the stability of meteorological models has to be explored by two dimensional models, similar to [9]. Here only the result of one-dimensional calculations is given. Since a very small time-step was used, time discretization does not do much to destroy energy conservation.

A one-dimensional model was obtained by putting  $v$  and all  $y$  derivatives to 0 in Eq. (9) and Eq. (21). The equations can be nondimensionalized using a space scale  $X_0$  and a time scale  $t_0$ . The finite element scheme was obtained by using a periodic channel with 22 node points being  $X_0$  apart, except for points 10 to 12, which were only  $\frac{1}{2}X_0$  apart. A gravity wave was obtained by using as initial condition  $u = 0$  and  $H = 1$  for  $x \in (5X_0, 10X_0)$  and  $H = .8$  for  $x \notin (5X_0, 10X_0)$ .

For comparison, a nonconserving scheme, referred to as the control scheme, was obtained by replacing  $G_1$  in Eq. (9) by  $G_2$ . The time step used was  $0.01t_0$ . Figure 2 gives the energy diagram for  $t \in (0, t_0)$  for the control scheme and schemes A and B. A better conservation of energy by schemes A and B is apparent. In this simple example no nonlinear instability occurs, even for very long integrations, for any scheme.

## 5. CONCLUSIONS

Two energy conserving Galerkin finite element schemes were obtained for the primitive meteorological equations. One of them, the B-scheme, allows the use of linear elements for all fields, and is therefore relatively easy to implement in large models.

## REFERENCES

1. A. ARAKAWA, *J. Comput. Phys.* **1**, 119 (1966).
2. G. FIX, *SIAM J. Appl. Math.* **29**, 371 (1975).
3. D. C. JESPERSEN, *J. Comput. Phys.* **16**, 383 (1974).
4. K. A. CLIFFE, *Int. J. Numer. Methods Fluids* **1**, 117 (1981).
5. J. STEPELER, *Int. J. Comput. Fluids* **6**, 241 (1979).
6. J. STEPELER, *J. Fluid. Mech.* **5**, 595 (1979).
7. D. M. BURRIDGE, J. STEPELER, AND R. STRÜFING, ECMWF Technical Report No. 54, Reading, U.K., 1986 (unpublished).
8. J. STEPELER, *J. Comput. Phys.* **22**, 295 (1976).
9. R. SADOURNY, *J. Atmos. Sci.* **32**, 680 (1975).
10. H. C. DAVIES, *Quart. J. Roy. Meteor. Soc.* **102**, 405 (1976).
11. R. LEE, P. GRESHO, S. CHAN, R. SANI, AND M. CULLEN, in *Finite Elements in Fluids*, Vol. 4, (Wiley, Chichester, 1982).

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